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Asymptotics of the orthogonal polynomials for the Szegő class with a polynomial weight

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To Barry Simon on the occasion of his 60th birthday

Abstract

Let p be a trigonometric polynomial, non-negative on the unit circle $\mathbb{T}$. We say that a measure σ on $\mathbb{T}$ belongs to the polynomial Szegő class, if $d\sigma(e^{i\theta}) = \sigma'_{\rm ac}(e^{i\theta})\, d\theta + d\sigma_s(e^{i\theta})$, σ_s is singular, and

$$\int_0^{2\pi} p(e^{i\theta}) \log \sigma'_{\rm ac}(e^{i\theta})\, d\theta > -\infty.$$

For the associated orthogonal polynomials $\{\varphi_n\}$, we obtain pointwise asymptotics inside the unit disc $\mathbb{D}$. Then we show that these asymptotics hold in L^2-sense on the unit circle. As a corollary, we get an existence of certain modified wave operators.

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0. Introduction

Let σ be a non-trivial Borel probability measure on the unit circle $\mathbb{T} = \{z : |z| = 1\}$. Consider polynomials $\{\varphi_n\}$ orthonormal with respect to σ,

$$\int_{\mathbb{T}} \varphi_n \overline{\varphi_m}\, d\sigma = \delta_{nm},$$

where δ_{nm} is the Kronecker's symbol. Sometimes, it is more convenient to work with monic orthogonal polynomials $\{\Phi_n\}$, $\Phi_n(z) = z^n + a_{n,n-1}z^{n-1} + \cdots + a_{n,0}$. These polynomials satisfy

$$\int_{\mathbb{T}} \Phi_n \overline{\Phi_m}\, d\sigma = c_n \delta_{nm}$$

with $c_n = \|\Phi_n\|_\sigma^2 = \int_{\mathbb{T}} |\Phi_n|^2\, d\sigma$.

It is well known [7,16] that polynomials $\{\Phi_n\}$ generate a sequence $\{\alpha_n\}$, $|\alpha_n| < 1$, of the so-called Verblunsky coefficients through the recurrence relations

$$\begin{cases} \Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n \Phi_n^*(z), \\ \Phi_{n+1}^*(z) = \Phi_n^*(z) - \alpha_n z \Phi_n(z), \end{cases}$$

where $\Phi_0(z) = 1$, $\Phi_0^*(z) = 1$, and $\Phi_n^*(z) = z^n \overline{\Phi_n(1/\bar{z})}$. Conversely, the measure σ (and polynomials $\{\varphi_n\}$) are completely determined by the sequence $\{\alpha_k\}$ of its Verblunsky parameters. Hence, it is natural to study properties of the sequence $\{\alpha_k\}$ and polynomials $\{\varphi_n\}$ in terms of σ and vice versa.

We say that σ is a Szegő measure ($\sigma \in$ (S), for brevity), if $d\sigma = d\sigma_{\rm ac} + d\sigma_s = \sigma'_{\rm ac} dm + d\sigma_s$ and the density $\sigma'_{\rm ac}$ of the absolutely continuous part of σ is such that

$$\int_{\mathbb{T}} \log \sigma'_{\rm ac}\, dm > -\infty.$$

Here, the singular part of σ is denoted by σ_s, and m is the probability Lebesgue measure on $\mathbb{T}$, $dm(t) = dt/(2\pi i t) = 1/(2\pi)\, d\theta$, $t = e^{i\theta} \in \mathbb{T}$.

The following theorem is classical:

Theorem 0.1 (*Geronimus [7], Szegő [18]*). *The following assertions are equivalent*:

(i) *the sequence $\{\alpha_k\}$ is in $\ell^2(\mathbb{Z}_+)$*,
(ii) *the measure σ belongs to the Szegő class*,
(iii) *analytic polynomials are not dense in $L^2(\sigma)$*.

We denote by $\mathcal{P}_0$ the set of analytic polynomials f such that $f \neq 0$ on $\mathbb{D}$ and $f(0) > 0$. Let also $\mathcal{P}_1 = \{f : f \in \mathcal{P}_0, f(0) = 1\}$. Then, the last statement of the theorem can be made more precise. Namely, we have [7,18] that

$$\begin{aligned} d(\mathcal{P}_1, 0)^2_{L^2(\sigma)} &= \inf_{f\in\mathcal{P}_1} \|f\|_\sigma^2 = \inf_{f \in \mathcal{P}_0,\, \|f\|_\sigma \leqslant 1} |f(0)|^{-2} \\ &= \exp \int_{\mathbb{T}} \log \sigma'_{\rm ac}\, dm. \end{aligned} \qquad (0.1)$$

If $\sigma \in \text{(S)}$, we define a function D, lying in the Hardy space $H^2(\mathbb{D})$ on the unit disk $\mathbb{D} = \{z : |z| < 1\}$, as

$$D(z) = \exp\left(\frac{1}{2}\int_{\mathbb{T}} \frac{t+z}{t-z} \log \sigma'_{\text{ac}}(t)\, dm(t)\right). \tag{0.2}$$

Theorem 0.2 (*Geronimus [7], Szegő [18]*). *Let* $\sigma \in \text{(S)}$. *Then*

$$\lim_{n\to\infty} D(z)\varphi_n^*(z) = 1$$

for every $z \in \mathbb{D}$. *Moreover*,

$$\lim_{n\to\infty} \int_{\mathbb{T}} |D\varphi_n^* - 1|^2\, dm = 0.$$

A modern presentation and recent advances in this direction can be found in [10,16].

It seems interesting to obtain similar results for different classes of measures. Consider a trigonometric polynomial p with the property $p(t) \geqslant 0$, $t \in \mathbb{T}$. Without loss of generality we can assume it is in the form

$$p(t) = \prod_{k=1}^{N} |t - \zeta_k|^{2\kappa_k}, \tag{0.3}$$

where $\{\zeta_k\}$ are points on $\mathbb{T}$ and $\kappa_k > 0$ are their "multiplicities". We say that σ is in the polynomial Szegő class (i.e., σ is a (pS)-measure or $\sigma \in \text{(pS)}$), if $d\sigma = \sigma'_{\text{ac}} dm + d\sigma_s$, σ_s being the singular part of the measure, and

$$\int_{\mathbb{T}} p(t) \log \sigma'_{\text{ac}}(t)\, dm(t) > -\infty. \tag{0.4}$$

The main result of the paper is a counterpart of Theorem 0.2 for orthogonal polynomials with respect to polynomial Szegő measures. We want to mention here that similar results for Jacobi matrices, including L^2-asymptotics of orthogonal polynomials on a segment, were obtained recently by Damanik and Simon [4]. The authors were able to deal with the case considered in [3, Theorem 3.1].

We introduce some notation. Actually, all objects appearing below should be indexed by the polynomial p from (0.3). We omit this dependence.

Let $\sigma \in \text{(pS)}$. Consider a modified Schwarz kernel

$$K(t,z) = \frac{t+z}{t-z}\,\frac{q(t)}{q(z)} = \frac{t+z}{t-z}\,\frac{q_0(t)}{q_0(z)}, \tag{0.5}$$

where $q_0(t) = \prod_{k=1}^{N}(t-\zeta_k)^{2\kappa_k}/t^{N'}$, $N' = \sum_k \kappa_k$, and $q(t) = Cq_0(t)$. The constant C equals $(\prod_k (-\zeta_k)^{\kappa_k})^{-1}$, so that $|C| = 1$ and $q(t) = \prod_k |t-\zeta_k|^{2\kappa_k} = p(t)$ for $t \in \mathbb{T}$. Let us introduce

$$\tilde{D}(z) = \exp\left(\frac{1}{2}\int_{\mathbb{T}} K(t,z) \log \sigma'_{\text{ac}}(t)\, dm(t)\right) \tag{0.6}$$

$$\tilde{\varphi}_n^*(z) = \exp\left(\int_{\mathbb{T}} K(t,z) \log |\varphi_n^*(t)|\, dm(t)\right) \tag{0.7}$$

with $z \in \mathbb{D}$. We call functions $\{\tilde{\varphi}_n^*\}$ the modified reversed polynomials. The properties of the kernel K easily imply that $|\tilde{D}|^2 = \sigma'_{\mathrm{ac}}$ and $|\tilde{\varphi}_n^*| = |\varphi_n^*|$ a.e. on $\mathbb{T}$, see Lemma 3.1. It is also useful to consider the functions

$$\psi_n(z) = \frac{\tilde{\varphi}_n^*(z)}{\varphi_n^*(z)} = \exp\left(\int_{\mathbb{T}} \frac{t+z}{t-z}\left(\frac{q(t)}{q(z)} - 1\right)\log|\varphi_n^*(t)|\,dm(t)\right).$$

Clearly $|\psi_n| = 1$ a.e. on $\mathbb{T}$ and, by (ii), Lemma 3.1,

$$\psi_n(z) = \exp\left\{A_0^{(n)} + \sum_{k=1}^{N}\sum_{j=1}^{2\kappa_n} A_{j,k}^{(n)}\left(\frac{z+\zeta_k}{z-\zeta_k}\right)^j\right\}, \tag{0.8}$$

where $A_0^{(n)}, A_{2j,k}^{(n)} \in i\mathbb{R}$ and $A_{2j+1,k}^{(n)} \in \mathbb{R}$. The coefficients $\{A_0^{(n)}, A_{j,k}^{(n)}\}$ can be expressed in a closed form through the Verblunsky coefficients $\{\alpha_k\}$.

The following theorem holds:

Theorem 0.3. *Let* $\sigma \in (\mathrm{pS})$. *Then*

$$\lim_{n\to\infty} \tilde{D}(z)\tilde{\varphi}_n^*(z) = 1$$

for every $z \in \mathbb{D}$.

The proof of the theorem is largely inspired by the classical proof of Theorem 0.2 [7,18], and it is based on appropriate sum rules. These sum rules are obtained in Theorem 2.3. Their proof is a translation of [14, Theorem 1.5], to the case of orthogonal polynomials on the unit circle. We also mention that the relations we prove in the theorem are closely related to sum rules obtained in [9,11–13]. A counterpart of Theorem 0.3 for Jacobi matrices is [14, Theorem 1.6].

A subsequent analysis shows that Theorem 0.3 can be considerably strengthened.

Theorem 0.4. *Let* $\sigma \in (\mathrm{pS})$. *Then*

$$\lim_{n\to\infty}\int_{\mathbb{T}} |\tilde{D}\tilde{\varphi}_n^* - 1|^2\,dm = 0.$$

The proof of the theorem is rather technical. One of the main observations leading to the statement is that

$$\lim_{n\to\infty}\int_{I} |\tilde{D}\tilde{\varphi}_n^* - 1|^2\,dm = 0$$

for any closed arc $I \subset \mathbb{T}$ that does not contain points $\{\zeta_k\}$. We prove the latter relation showing that

$$|\tilde{D}\tilde{\varphi}_n^*(z)| \leqslant \frac{C_\varepsilon}{\sqrt{1-|z|}}$$

for $z \in \mathbb{D}\backslash(\cup_k B_\varepsilon(\zeta_k))$, $B_\varepsilon(\zeta) = \{z : |z-\zeta| < \varepsilon\}$, whenever $\varepsilon > 0$ is small enough. It is crucial that the above constant C_ε does not depend on n.

We apply Theorem 0.4 to construct modified wave operators for the CMV-representations $\mathcal{C}, \mathcal{C}_0$ associated to measures $\sigma \in$ (pS) and m, see Section 1 for the definitions and notation. For the Szegő case, the classical wave operators were described recently by Simon [16, Section 10.7]. Let $\mathcal{F}_0 : L^2(m) \to \ell^2(\mathbb{Z}_+)$, $\mathcal{F} : L^2(\sigma) \to \ell^2(\mathbb{Z}_+)$ be the Fourier transforms related to $\mathcal{C}$ and $\mathcal{C}_0$. Recall that

$$\mathcal{C} = \mathcal{F} z \mathcal{F}^{-1}, \quad \mathcal{C}_0 = \mathcal{F}_0 z \mathcal{F}_0^{-1}.$$

Theorem 0.5. *Let $\sigma \in$ (pS). The limits*

$$\tilde{\Omega}_\pm = \text{s-lim}_{n\to\pm\infty} e^{W(\mathcal{C},2n)} \mathcal{C}^n \mathcal{C}_0^{-n} \tag{0.9}$$

exist. Here

$$W(\mathcal{C}, n) = A_0^{(n)} + \sum_{k=1}^{N} \sum_{j=1}^{2\kappa_k} A_{j,k}^{(n)} \left(\frac{\mathcal{C} + \zeta_k}{\mathcal{C} - \zeta_k} \right)^j$$

and coefficients $\{A_0^{(n)}, A_{j,k}^{(n)}\}$ are defined in (0.8). *Furthermore*,

$$\mathcal{F}^{-1} \tilde{\Omega}_+ \mathcal{F}_0 = \chi_{E_{\text{ac}}} \frac{1}{\tilde{D}}, \quad \mathcal{F}^{-1} \tilde{\Omega}_- \mathcal{F}_0 = \chi_{E_{\text{ac}}} \frac{1}{\bar{\tilde{D}}}, \tag{0.10}$$

where $E_{\text{ac}} = \mathbb{T} \backslash \operatorname{supp} \sigma_s$.

In the formulation above, s-lim refers to the limit in $\ell^2(\mathbb{Z}_+)$ in the strong sense. A natural problem is to pass from wave operators (0.9) to operators of the form

$$\text{s-lim}_{n\to\pm\infty} \mathcal{C}^n \mathcal{C}_0^{-n} e^{\tilde{W}(\mathcal{C}_0,n)}.$$

This question is still open, see [2] in this connection.

Finally, we address a variational principle that is naturally connected to measures from a (pS)-class. Let p be the trigonometric polynomial from (0.3). We pick a constant C_0 in a way that $C_0 \int_{\mathbb{T}} p\, dm = 1$, and let $p_0 = C_0 p$.

For a $g \in \mathcal{P}_0$, we define

$$\lambda(g) = \exp\left(\int_{\mathbb{T}} p_0 \log |g|\, dm \right)$$

and $\mathcal{P}_1' = \{g : g \in \mathcal{P}_0,\ \lambda(g) = 1\}$.

Theorem 0.6. *Let $d\sigma = \sigma_{\text{ac}}' \, dm + d\sigma_s$. Then*

$$\exp\left(\int_{\mathbb{T}} p_0 \log \frac{\sigma_{\text{ac}}'}{p_0}\, dm \right) \leqslant \inf_{g\in\mathcal{P}_1'} \|g\|_\sigma^2 = \inf_{\substack{g\in\mathcal{P}_0,\\ \|g\|_\sigma \leqslant 1}} \frac{1}{|\lambda(g)|^2}$$
$$\leqslant \exp\left(\int_{\mathbb{T}} p_0 \log \sigma_{\text{ac}}'\, dm \right). \tag{0.11}$$

Remind that σ is a Szegő measure if and only if the system $\{e^{iks}\}_{k\in\mathbb{Z}}$ is uniformly minimal in $L^2(\sigma)$ [6, Chapter 3; 15, Chapter 6]. Saying that σ is a (pS)-measure translates into the uniform minimality of another system, $\{e^{ikv(s)}\}_{k\in\mathbb{Z}}$, in the same space $L^2(\sigma)$. Above,

$$v(s) = \int_0^s p_0(e^{is'})\, ds',$$

where $s, s' \in [0, 2\pi]$; see [14], Lemma 2.2.

We now turn to the concrete example to illustrate our results. It was proved recently in [16, Section 2.8] that $\sigma \in (\mathrm{p_1S})$ with

$$p_1(t) = \frac{1}{2}\,|1-t|^2 = 1 - \cos\theta$$

if and only if $\{\alpha_k\} \in \ell^4(\mathbb{Z}_+)$ and $\{\alpha_{k+1} - \alpha_k\} \in \ell^2(\mathbb{Z}_+)$ (above, $t = e^{i\theta}$). This class of parameters was studied earlier in [5]. Theorems 0.3–0.6 readily apply to this special case. In particular, we have

$$K_1(t,z) = \frac{t+z}{t-z}\,\frac{(t-1)^2}{t}\,\frac{z}{(1-z)^2} = -\frac{z}{(1-z)^2}\,\frac{t+z}{t-z}\,|1-t|^2,$$
$$\tilde{D}_1(z) = \exp\left(\frac{1}{2}\int_{\mathbb{T}} K_1(t,z)\log\sigma'_{\mathrm{ac}}(t)\,dm(t)\right)$$

and

$$\psi_n(z) = \exp\left(A_n\,\frac{1+z}{1-z} + B_n\left\{\left(\frac{1+z}{1-z}\right)^2 - 1\right\}\right),$$

where

$$A_n = \sum_{k=0}^{n}\log(1-|\alpha_k|^2)^{1/2}, \quad B_n = \frac{i}{4}\,\mathrm{Im}\left(\alpha_0 - \sum_{k=1}^{n}\bar{\alpha}_{k-1}\alpha_k\right).$$

Recently, for $t_1, t_2 \in \mathbb{T}$, the following class of polynomials p was considered [17]

$$p(t) = |(t-t_1)(t-t_2)|^2$$

and the criteria for (0.4) to be true were obtained in terms of the Verblunsky coefficients. Methods of the current paper are also applicable to this case.

We conjecture that counterparts of Theorems 0.4, 0.5 hold true for Jacobi matrices; see [9,14] in this connection.

The paper is organized as follows. The preliminaries are in Section 1. The sum rules we use in the proof of Theorem 0.3 are obtained in Section 2. Theorem 0.3 itself is proved in Section 3, and it is "upgraded" to the asymptotics in $L^2(\mathbb{T})$-sense in Section 4. Section 5 deals with the modified wave operators and the variational principle from Theorem 0.6.

As usual, $H^p(\mathbb{D})$ is the Hardy space of analytic functions on the unit disk [6]. For an arc $I \subset \mathbb{T}$, we write $L^2(I)$ to refer to the standard L^2-space with the Lebesgue measure on I. We set $\log^+ x = (|\log x| + \log x)/2$ and $\log^- x = (|\log x| - \log x)/2$ for $x > 0$. Also, C is a constant changing from one relation to another.

1. Preliminaries

It is useful to keep in mind the simple general properties of the measures from a (pS)-class. Following [10,16], we say that σ belongs to the Erdős class ($\sigma \in$ (E)) if $\sigma'_{ac} > 0$ a.e. on $\mathbb{T}$. A measure is in the Nevai class ($\sigma \in$ (N)) if $\lim_{n\to\infty} \alpha_n = 0$. Lastly, σ is a Rakhmanov measure (i.e., $\sigma \in$ (R)) if

$$\text{w-}\lim_{n\to\infty} |\varphi_n|^2\, d\sigma = dm. \tag{1.1}$$

The following relations are true [10, Sections 2, 6, 7; 16, Sections 9.2, 9.3]

$$\text{(S)} \subset \text{(pS)} \subset \text{(E)} \subset \text{(N)} \subset \text{(R)}. \tag{1.2}$$

Here, the first and the second inclusions are obvious.

Let us recall a few facts on the so-called CMV-representations. More information on the topic can be found in [1,16, Section 2.3].

Let σ be a measure on $\mathbb{T}$. Consider the unitary operator $U : L^2(\sigma) \to L^2(\sigma)$ given by the formula $Uf(t) = tf(t)$, $f \in L^2(\sigma)$. It turns out one can find an orthonormal basis $\{\chi_n\}_{n\in\mathbb{Z}}$ in $L^2(\sigma)$ such that the matrix of U in this basis has a reasonably simple form. Namely, we set for $n \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$

$$\chi_n(z) = \begin{cases} z^{-k}\varphi^*_{2k}(z), & n = 2k, \\ z^{-(k-1)}\varphi_{2k-1}(z), & n = 2k-1. \end{cases}$$

Theorem 1.1 (*Cantero et al. [1], Simon [16, Section 2.3]*). *The operator U, defined above, is unitarily equivalent to the operator $\mathcal{C} : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ of the form*

$$\mathcal{C} = \mathcal{C}(\sigma) = \begin{bmatrix} * & * & * & 0 & 0 & \ldots \\ * & * & * & 0 & 0 & \ldots \\ 0 & * & * & * & * & \ldots \\ 0 & * & * & * & * & \ldots \\ 0 & 0 & 0 & * & * & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} A_0 & 0 & \ldots \\ 0 & A_1 & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix},$$

where $\alpha = \{\alpha_k\}$ is the sequence of Verblunsky coefficients of σ,

$$A_j = \begin{bmatrix} \bar\alpha_{k+1}\rho_k & -\bar\alpha_{k+1}\alpha_k & \bar\alpha_{k+2}\rho_{k+1} & \rho_{k+2}\rho_{k+1} \\ \rho_{k+1}\rho_k & -\rho_{k+1}\alpha_k & -\bar\alpha_{k+2}\alpha_{k+1} & -\rho_{k+2}\alpha_{k+1} \end{bmatrix},$$

$$A_0 = \begin{bmatrix} \bar\alpha_0 & \bar\alpha_1\rho_0 & \rho_1\rho_0 \\ \rho_0 & -\bar\alpha_1\alpha_0 & -\rho_1\alpha_0 \end{bmatrix}$$

and $\rho_k = (1 - |\alpha_k|^2)^{1/2}$.

The matrix $\mathcal{C}$ is called a CMV-representation associated to the measure σ.

It is easy to see that the map $\mathcal{F} : L^2(\sigma) \to \ell^2(\mathbb{Z}_+)$, carrying out the unitary equivalence

$$\mathcal{C} = \mathcal{F}z\mathcal{F}^{-1}.$$

is determined by relations $(\mathcal{F}f)_n = \int_{\mathbb{T}} f\bar{\chi}_n\, d\sigma$, where $f \in L^2(\sigma)$. Similar objects, related to the Lebesgue measure m, are indexed by 0. That is, its CMV-matrix is denoted by $\mathcal{C}_0$, $\{\chi_n^{(0)}\}$ and $\mathcal{F}_0$ are the standard basis $\{t^k\}_{k\in\mathbb{Z}}$ and the Fourier transform, respectively.

Now, we denote by $\mathcal{C}_n$ the $n \times n$ upper left block of $\mathcal{C}$. One can prove [1], Theorem 3.1, [16], Theorems 1.7.18 and 4.2.47, that

$$\varphi_n(z) = \frac{1}{A_n}\det(z - \mathcal{C}_n), \quad \varphi_n^*(z) = \frac{1}{A_n}\det(1 - z\bar{\mathcal{C}}_n)(1 - z\bar{\mathcal{C}}_{0,n})^{-1}$$

with $A_n = \prod_{k=0}^{n-1}\rho_k$. Recalling definition (0.2), we get the following theorem:

Theorem 1.2 (*Cantero et al. [1], Simon [16]*). *Let* $\sum_k |\alpha_k| < \infty$. *Then, for* $z \in \mathbb{D}$,

$$D(z) = A_\infty \det(1 - z\bar{\mathcal{C}}_0)(1 - z\bar{\mathcal{C}})^{-1}$$

Moreover, we have $\log D(z) = t_0 + \sum_{k=1}^{\infty}(t_k/k)\, z^k$ *and*

$$t_0 = \sum_k \log\rho_k = \sum_k \log(1 - |\alpha_k|^2)^{1/2}, \quad t_k = \operatorname{tr}(\bar{\mathcal{C}}^k - \bar{\mathcal{C}}_0^k) \tag{1.3}$$

with $k \geqslant 1$.

2. Polynomial Szegő condition and corresponding sum rules

We fix the polynomial p (0.3) for the rest of this paper.

The goal of this section is to obtain the sum rules similar to [16, Section 2.8; 14, Theorem 1.5]. With the exception of simple technical details, our argument follows word-by-word a reasoning from [14].

We start with a CMV-representation $\mathcal{C}$ having the property rank $(\mathcal{C} - \mathcal{C}_0) < \infty$. Note that this is equivalent to saying that the sequence of Verblunsky coefficients $\{\alpha_k\}$, corresponding to $\mathcal{C}$, is finite. Therefore, $\sum_k |\alpha_k| < \infty$ and, by Theorem 1.2,

$$\log D(z) = t_0 + \sum_{k=1}^{\infty}\frac{t_k}{k} z^k$$

with coefficients $\{t_k\}$ given by (1.3). Since $\log D \in H^\infty(\mathbb{D}) \cap C(\bar{\mathbb{D}})$, this yields

$$\int_{\mathbb{T}} \log|D|^2\, dm = 2t_0, \quad \int_{\mathbb{T}} t^k \log|D|^2\, dm = \frac{\bar{t}_k}{k}. \tag{2.1}$$

Taking polynomial p from (0.3), we define an analytic polynomial P through the relations

$$p_1 = 2P_+(p), \quad P'(t) = \frac{p_1(t) - p_1(0)}{t}, \quad P(0) = 0 \tag{2.2}$$

here $P_+ : L^2(\mathbb{T}) \to H^2(\mathbb{D})$ is the Riesz projection [6, Chapter 3].

Lemma 2.1. *Let p be as above and* rank $(\mathcal{C} - \mathcal{C}_0) < \infty$. *We have*

$$\int_{\mathbb{T}} p(t) \log |D(t)|^2 \, dm(t) = A_0 t_0 + \operatorname{Re} \operatorname{tr}(P(\mathcal{C}) - P(\mathcal{C}_0)), \tag{2.3}$$

where $A_0 = p_1(0) = 2 \int_{\mathbb{T}} p(t) \, dm$.

Proof. Write p as $p(t) = a_0 + 2\mathrm{Re} \sum_{j=1}^N a_j t^j$. Recalling (2.1), we get

$$\begin{aligned}\int_{\mathbb{T}} p \log |D|^2 \, dm &= 2a_0 t_0 + 2\mathrm{Re} \sum_{j=1}^N a_j \int_{\mathbb{T}} t^j \log |D|^2 \, dm \\ &= 2a_0 t_0 + 2\mathrm{Re} \sum_{j=1}^N \frac{a_j}{j} \bar{t}_j = 2a_0 t_0 + 2\mathrm{Re} \sum_{j=1}^N \frac{a_j}{j} \operatorname{tr}(\mathcal{C}^j - \mathcal{C}_0^j).\end{aligned}$$

It remains to notice that the polynomial $2 \sum_j (a_j/j) z^j$ above is indeed P given by (2.2) and $A_0 = 2a_0$. Hence, the last expression in the displayed formula is exactly $A_0 t_0 + \operatorname{Re} \operatorname{tr}(P(\mathcal{C}) - P(\mathcal{C}_0))$, and the lemma is proved. □

Let us set

$$\begin{aligned}\Phi(\mathcal{C}) &= \int_{\mathbb{T}} p \log \sigma'_{\mathrm{ac}} \, dm \\ \Psi(\mathcal{C}) &= A_0 t_0 + \operatorname{Re} \operatorname{tr}(P(\mathcal{C}) - P(\mathcal{C}_0)).\end{aligned}$$

Notice that $\Phi(\mathcal{C})$ is exactly the left-hand side of equality (2.3).

We now rewrite $\Psi(\mathcal{C})$ in a different form. Since $t_0 = \sum \log \rho_k$, we have

$$\Psi(\mathcal{C}) = \sum_{k=0}^{\infty} \{A_0 \log \rho_k + \operatorname{Re}((P(\mathcal{C}) - P(\mathcal{C}_0)) e_k, e_k)\} \tag{2.4}$$

here $\{e_k\}$ is the standard basis in $\ell^2(\mathbb{Z}_+)$. Consider the shift $S : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ given by $Se_k = e_{k+1}$. For a bounded operator A on $\ell^2(\mathbb{Z}_+)$, take $\tau(A) = S^* A S$. It is obvious that the matrix of $\tau^k(A)$ is obtained from the matrix of A by dropping its first k rows and columns.

Furthermore, the degree of the polynomial P is N, the matrix $\mathcal{C}$ is five-diagonal, so $P(\mathcal{C})$ contains $4N + 1$ non-zero diagonals. Consequently, equality (2.4) is exactly the same as

$$\Psi(\mathcal{C}) = \sum_{k=0}^{2N+1} \{A_0 \log \rho_k + \operatorname{Re}((P(\mathcal{C}) - P(\mathcal{C}_0) e_k, e_k)\} + \sum_{k=0}^{\infty} \psi \circ \tau^k(\mathcal{C}),$$

where

$$\psi(\mathcal{C}) = A_0 \log \rho_{2N+2} + ((P(\mathcal{C}) - P(\mathcal{C}_0)) e_{2N+2}, e_{2N+2})$$

is a function of a finite number of Verblunsky coefficients.

The following lemma is similar to [14, Corollary 3.3].

Lemma 2.2 (*Nazarov et al. [14]*). *There exists a function γ depending on $l = 4N$ arguments such that*

$$\psi(x_1, \dots, x_{l+1}) = \eta(x_1, \dots, x_{l+1}) - \gamma(x_2, \dots, x_{l+1}) + \gamma(x_1, \dots, x_l)$$

and $\eta(x_1, \dots, x_{l+1}) \leqslant 0$ for any collection $(x_1, \dots, x_{l+1})$.

The proof of the lemma relies on the fact that $\Psi(\mathcal{C}) \leqslant C < \infty$ for all $\mathcal{C}$ with the property rank $(\mathcal{C} - \mathcal{C}_0) < \infty$. This is obviously true because we have $\Psi(\mathcal{C}) = \Phi(\mathcal{C})$ for these $\mathcal{C}$ and $\Phi(\mathcal{C})$ is uniformly bounded away from ∞ by the Jensen inequality. Now, define

$$\tilde{\Psi}(\mathcal{C}) = \sum_{k=0}^{2N+1} \{A_0 \log \rho_k + \mathrm{Re}((P(\mathcal{C}) - P(\mathcal{C}_0)e_k, e_k)\} + \sum_{k=0}^{\infty} \eta \circ \tau^k(\mathcal{C}) + \gamma(\mathcal{C}). \tag{2.5}$$

Theorem 2.3 (*Nazarov et al. [14]*). *A measure σ lies in the* (pS)*-class* (*see* (0.4)) *if and only if $\tilde{\Psi}(\mathcal{C}) > -\infty$, or, equivalently, $\sum_{k=0}^{\infty} \eta \circ \tau^k(\mathcal{C}) > -\infty$. Moreover,*

$$\Phi(\mathcal{C}) = \tilde{\Psi}(\mathcal{C}) = \Psi(\mathcal{C}) \tag{2.6}$$

The proof literally follows [14, Theorem 1.5], and it is close in spirit to arguments from [9,16, Section 2.8]. Its main ingredients are the non-positivity of η in (2.5) and the fact that $\lim_{k\to\infty} \alpha_k = 0$.

3. Pointwise asymptotics for orthogonal polynomials on the unit disk

We start with the following lemma:

Lemma 3.1. *Let $\sigma \in$* (pS), *the polynomials $\tilde{\varphi}_n^*$ and the function $\tilde{D}$ be defined in* (0.7), (0.6). *Then*

(i) $|\tilde{D}(t)|^2 = \sigma'_{\mathrm{ac}}(t)$ *a.e. on* $\mathbb{T}$,
(ii) $\tilde{\varphi}_n^* = \psi_n \varphi_n^*$ *and* $|\psi_n(t)| = 1$ *a.e. on* $\mathbb{T}$. *Moreover,*

$$\psi_n(z) = \exp\left(A_0^{(n)} + \sum_k \sum_{j=1}^{2\kappa_n} A_{j,k}^{(n)} \left\{ \frac{z + \zeta_k}{z - \zeta_k} \right\}^j \right), \tag{3.1}$$

where $A_0^{(n)},\ A_{2j,k}^{(n)} \in i\mathbb{R}$ and $A_{2j+1,k}^{(n)} \in \mathbb{R}$.

Proof. To prove claim (i), observe that

$$\log |\tilde{D}(z)|^2 = \mathrm{Re} \int_{\mathbb{T}} \frac{t + z}{t - z} \frac{q(t)}{q(z)} \log \sigma'_{\mathrm{ac}}(t)\, dm(t).$$

Also, $\operatorname{Im}\{q(t)/q(z)\}$ tends uniformly to 0 as z goes to $\mathbb{T}\backslash\{\zeta_k\}$, and $q=\operatorname{Re} q=p$ on $\mathbb{T}$. Consequently, for a.e. $t_0\in\mathbb{T}\backslash\{\zeta_k\}$,

$$\begin{aligned}\lim_{z\to t_0}\log|\tilde D(z)|^2&=\lim_{z\to t_0}\frac{1}{\operatorname{Re} q(z)}\int_{\mathbb{T}}\operatorname{Re}\frac{t+z}{t-z}\operatorname{Re} q(t)\log\sigma'_{\mathrm{ac}}(t)\,dm(t)\\&=\frac{1}{p(t_0)}\,p(t_0)\log\sigma'_{\mathrm{ac}}(t_0),\end{aligned}$$

where we used the standard properties of the Poisson kernel $\operatorname{Re}(t+z)/(t-z)$.

The computation also shows that $|\tilde\varphi_n^*|=|\varphi_n^*|$ a.e. on $\mathbb{T}$, and, in particular, $|\psi_n|=1$ a.e. On the other hand,

$$\psi_n(z)=\exp\left(\int_{\mathbb{T}}\left\{\frac{t+z}{t-z}\,\frac{q(t)-q(z)}{q(z)}\right\}\log|\varphi_n^*(t)|\,dm(t)\right).$$

The function in the curled brackets is rational with respect to z and its degree is $2N'$, $N'=\sum_k\kappa_k$. Its poles have multiplicities $2\kappa_k$ and they are located at $\{\zeta_k\}$. So, we get

$$\frac{t+z}{t-z}\,\frac{q(t)-q(z)}{q(z)}=a_0(t)+\sum_k\sum_{j=1}^{2\kappa_k}a_{jk}(t)\left(\frac{z+\zeta_k}{z-\zeta_k}\right)^j,$$

where a_0,a_{jk} are some trigonometric polynomials (i.e., polynomials with respect to $t,\bar t$). We now put $A_0^{(n)}=\int_{\mathbb{T}}a_0\log|\varphi_n^*|\,dm$, $A_{j,k}^{(n)}=\int_{\mathbb{T}}a_{jk}\log|\varphi_n^*|\,dm$, and recall that $|\psi_n|=1$ a.e. on $\mathbb{T}$. Hence, the function under the exponent sign in (3.1) is purely imaginary a.e. on $\mathbb{T}$. This implies the properties of $\{A_0^{(n)},A_{j,k}^{(n)}\}$ stated in the lemma, and the proof of (ii) is completed. $\square$

The formulas for coefficients $\{A_0^{(n)},A_{j,k}^{(n)}\}$ in terms of the CMV-representation can be obtained with the help of the map $p\mapsto P$ described in (2.2); the only difference is that $\mathcal{C}$ should be replaced with its $n\times n$ upper left block $\mathcal{C}_n$.

Proof of Theorem 0.3. We pick a constant C_1 in a way that $0\leqslant C_1p\leqslant 1$ on $\mathbb{T}$. It is convenient to define

$$\begin{aligned}f_n(z)&=\exp\left(\frac12\int_{\mathbb{T}}\frac{t+z}{t-z}\log\alpha_n(t)\,dm\right),\\f(z)&=\exp\left(\frac12\int_{\mathbb{T}}\frac{t+z}{t-z}\log\alpha(t)\,dm\right)\end{aligned}$$

with $\alpha_n(t)=(|\varphi_n^*(t)|^{-2})^{C_1p(t)}$, $\alpha(t)=\sigma'_{\mathrm{ac}}(t)^{C_1p(t)}$, and $z\in\mathbb{D}$. Obviously, it is enough to show that $\lim_{n\to\infty}\log f_n(z)=\log f(z)$. Recalling $\int_{\mathbb{T}}|\varphi_n^*|^{-2}\,dm=1$ [16, Theorem 1.7.8], we have

$$\begin{aligned}\int_{\mathbb{T}}|f_n|^2\,dm=\int_{\mathbb{T}}\left(\frac{1}{|\varphi_n^*|^2}\right)^{C_1p}dm&=\int_{|\varphi_n^*|^{-2}\leqslant 1}\left(\frac{1}{|\varphi_n^*|^2}\right)^{C_1p}dm\\&\quad+\int_{|\varphi_n^*|^{-2}\geqslant 1}\left(\frac{1}{|\varphi_n^*|^2}\right)^{C_1p}dm\leqslant 2.\end{aligned}$$

It follows similarly that $\int_{\mathbb{T}} |f|^2\, dm < \infty$. So, the functions f_n, f are outer and $\{f_n\}$ is uniformly bounded in $H^2(\mathbb{D})$. A ball in $H^2(\mathbb{D})$ is weakly compact, and weak convergence implies the pointwise convergence on $\mathbb{D}$. Consequently, there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ that converges to a function $f_0 \in H^2(\mathbb{D})$ in $\mathbb{D}$.

We now prove that $f_0 = f$. Indeed, for a $z \in \mathbb{D}$

$$\limsup_n \frac{1}{2}\int_{\mathbb{T}} \operatorname{Re}\frac{t+z}{t-z}\, p(t) \log \frac{1}{|\varphi_n^*(t)|^2}\, dm(t) \leqslant \frac{1}{2}\int_{\mathbb{T}} \operatorname{Re}\frac{t+z}{t-z}\, p(t) \log \sigma'_{\mathrm{ac}}(t)\, dm(t).$$

Here, we kept in mind that the measures $|\varphi_n^*|^{-2} dm$ tend weakly to σ and the above expressions are semicontinuous with respect to this type of convergence [9, Section 5; 16, Section 2.3]. This implies that $|f_0(z)| \leqslant |f(z)|$ for all $z \in \mathbb{D}$. We also observe that

$$\log f_n(0) = \frac{1}{2}\int_{\mathbb{T}} (C_1 p) \log \frac{1}{|\varphi_n^*|^2}\, dm = \frac{1}{2} C_1 \tilde{\Psi}(\mathcal{C}_n),$$

where $\tilde{\Psi}$ is an expression from (2.5) and $\mathcal{C}_n$ is the truncated CMV-matrix. Identity (2.6) from Theorem 2.3 reads as $\log f(0) = (1/2) C_1 \tilde{\Psi}(\mathcal{C})$. In particular, we have

$$\lim_{n\to\infty} \tilde{\Psi}(\mathcal{C}_n) = \tilde{\Psi}(\mathcal{C}) \tag{3.2}$$

which is equivalent to

$$\log f_0(0) = \lim_{k\to\infty} \log f_{n_k}(0) = \log f(0).$$

Since the function f is outer, $|f_0| \leqslant |f|$ and $|f_0(0)| = |f(0)|$, the usual multiplicative representation of the functions from $H^2(\mathbb{D})$ imply $f = f_0$ on $\mathbb{D}$. Thus, the sequence $\{f_n\}$ itself converges to the function f, and the theorem is proved. □

Remark 3.2. Since the function η in (2.5) is non-positive, the convergence in (3.2) is monotone, and $f_{n+1}(0) \leqslant f_n(0)$ for large n.

4. Asymptotics of orthogonal polynomials in L^2-sense

For any $\varepsilon > 0$, let $B_\varepsilon[\zeta] = \{z : |z-\zeta| \leqslant \varepsilon\}$. Furthermore, let $\Omega_\varepsilon = \mathbb{D} \backslash (\cup_k B_\varepsilon[\zeta_k])$, $I_{k,\varepsilon} = \mathbb{T} \cap B_\varepsilon[\zeta_k]$, and $A_\varepsilon = \cup_k I_{k,\varepsilon}$. We need several lemmas to prove the main theorem of this section.

Lemma 4.1. *Let $\sigma \in$ (pS). Then, for a finite union of intervals $E \subset \mathbb{T}$*

$$\limsup_n \int_E p |\log(|\varphi_n^*|^2 \sigma'_{\mathrm{ac}})|\, dm < \infty.$$

Proof. We start by proving that

$$\limsup_n \int_E p \log^+(|\varphi_n^*|^2 \sigma'_{\mathrm{ac}})\, dm < \infty.$$

Indeed, by $\log^+ x \leqslant x$, $x > 0$ we get

$$\int_E p \log^+(|\varphi_n^*|^2 \sigma'_{\mathrm{ac}})\, dm \leqslant C \int_E \log^+(|\varphi_n^*|^2 \sigma'_{\mathrm{ac}})\, dm \leqslant C \int_E |\varphi_n|^2 \sigma'_{\mathrm{ac}}\, dm$$
$$\leqslant C \int_{\mathbb{T}} |\varphi_n|^2\, d\sigma = C.$$

To show that

$$\limsup_n \int_E p \log^-(|\varphi_n^*|^2 \sigma'_{\mathrm{ac}})\, dm < \infty$$

it suffices to know

$$\liminf_n \int_E p \log(|\varphi_n^*|^2 \sigma'_{\mathrm{ac}})\, dm > -\infty.$$

We have that the measures $\{|\varphi_n^*|^{-2}\, dm\}$ tend weakly to $d\sigma$, and by the semicontinuity of the entropy [9, Section 5; 16, Section 2.3]

$$\limsup_n \int_E p \log \frac{1}{|\varphi_n^*|^2}\, dm \leqslant \int_E p \log \sigma'_{\mathrm{ac}}\, dm.$$

Consequently,

$$\liminf_n \int_E p \log(|\varphi_n^*|^2 \sigma'_{\mathrm{ac}})\, dm \geqslant 0.$$

The lemma is proved. $\square$

Lemma 4.2. *Let* $\sigma \in (\mathrm{pS})$ *and*

$$\xi_n(z) = \tilde{D}(z)\tilde{\varphi}_n^*(z) = \exp\left(\frac{1}{2}\int_{\mathbb{T}} K(t,z) \log(|\varphi_n^*|^2 \sigma'_{\mathrm{ac}})\, dm\right). \tag{4.1}$$

Then, for $z \in \Omega_{2\varepsilon}$

$$|\xi_n(z)| \leqslant \frac{C_\varepsilon}{\sqrt{1-|z|}},$$

where the constant C_ε *does not depend on* n.

Proof. We get $\xi_n = f'_n f''_n$ with

$$f'_n(z) = \exp\left(\frac{1}{2}\int_{A_\varepsilon} K(t,z) \log(|\varphi_n^*|^2 \sigma'_{\mathrm{ac}})\, dm\right),$$
$$f''_n(z) = \exp\left(\frac{1}{2}\int_{\mathbb{T}\setminus A_\varepsilon} K(t,z) \log(|\varphi_n^*|^2 \sigma'_{\mathrm{ac}})\, dm\right).$$

It is plain that, for $t \in A_\varepsilon$, $z \in \Omega_{2\varepsilon}$, the expressions $|(t+z)/(t-z)|$, $1/|q(z)|$ are bounded by constants depending on ε. Lemma 4.1 shows that

$$\limsup_n \int_{A_\varepsilon} p|\log(|\varphi_n^*|^2 \sigma'_{\mathrm{ac}})|\, dm < \infty$$

and, therefore, $|f_n'(z)| \leqslant C$ for $z \in \Omega_{2\varepsilon}$. Passing to f_n'', we represent it as

$$\begin{aligned} f_n''(z) &= \exp\left(\frac{1}{2}\int_{\mathbb{T}} K(t,z)\log\beta_n(t)\,dm\right) \\ &= \exp\left(\frac{1}{2}\int_{\mathbb{T}} \frac{t+z}{t-z}\left(\frac{q(t)}{q(z)}-1\right)\log\beta_n(t)\,dm\right) \\ &\quad\times\exp\left(\frac{1}{2}\int_{\mathbb{T}} \frac{t+z}{t-z}\log\beta_n(t)\,dm\right) = g_n'(z)g_n''(z), \end{aligned} \tag{4.2}$$

where

$$\beta_n(t) = \begin{cases} 1, & t \in A_\varepsilon, \\ |\varphi_n^*|^2\sigma_{\mathrm{ac}}', & t \in \mathbb{T}\backslash A_\varepsilon. \end{cases}$$

Once again, Lemma 4.1 implies that

$$\limsup_n \int_{\mathbb{T}} p|\log\beta_n|\,dm < \infty.$$

Since $0 < c \leqslant p(t) \leqslant C$ for $t \in \mathbb{T}\backslash A_\varepsilon$, we get

$$\limsup_n \int_{\mathbb{T}} |\log\beta_n(t)|\,dm(t) < \infty.$$

Furthermore,

$$\left|\frac{t+z}{t-z}\,\frac{q(t)-q(z)}{q(z)}\right| \leqslant C$$

for all $z \in \Omega_{2\varepsilon}$, and we obtain that $|g_n'(z)| \leqslant C$.

The functions g_n'' lie in the Nevanlinna class and are outer. Moreover, we have

$$\int_{\mathbb{T}} |g_n''|^2\,dm = \int_{\mathbb{T}} \beta_n\,dm = \int_{\mathbb{T}\backslash A_\varepsilon} |\varphi_n^*|^2\sigma_{\mathrm{ac}}'\,dm + m(A_\varepsilon) \leqslant C$$

so $g_n'' \in H^2(\mathbb{D})$ and $\|g_n''\|_2 \leqslant C$. To finish the proof of the lemma, we invoke a standard argument (the integral Cauchy formula or properties of the reproducing kernel in $H^2(\mathbb{D})$)

$$|g_n''(z)| = \left|\left(g_n'', \frac{1}{1-\bar{z}t}\right)\right| \leqslant \|g_n''\|_2 \left\|\frac{1}{1-\bar{z}t}\right\|_2 \leqslant \frac{C}{\sqrt{1-|z|^2}}. \qquad \square$$

The proof of the following lemma is close in spirit to [8, Lemma 3.2].

Lemma 4.3. *Let* $\sigma \in (\mathrm{pS})$. *Then*

$$\lim_{n\to\infty} \int_{I'} |\tilde{D}\tilde{\varphi}_n^* - 1|^2\,dm = 0,$$

where I' *is any closed arc on* $\mathbb{T}$ *which does not contain any point* $\{\zeta_k\}$.

Proof. We fix any closed arc I which does not contain any $\{\zeta_k\}$ and such that $I' \subset I$. As before, $\xi_n = \tilde{D}\tilde{\varphi}_n^*$.

Let Ω be the shaded domain on Fig. 1. Let also $u : \mathbb{D} \to \Omega$ and $v : \Omega \to \mathbb{D}$ be mutually inverse conformal maps of the domains, that is, $u(v(\zeta)) = \zeta$ and $v(u(z)) = z$ for $z \in \mathbb{D}$, $\zeta \in \Omega$. We set $\partial\Omega$ to be the boundary of Ω, $\partial\Omega = I \cup I_1 \cup I_2$, where I is the arc on $\mathbb{T}$ and I_1, I_2 are the straight segments, see the figure. The angles between I, I_1 and I_2 are π/α, $\alpha > 1$. Furthermore, let $\zeta_0 = u(0) \in \Omega$ and η_1, η_2 be the "corners" of Ω. It is plain that, for $i = 1, 2$

(i) There are constants $c, C > 0$ such that

$$c|\zeta - \eta_i|^\alpha \leqslant |v(\zeta) - v(\eta_i)| \leqslant C|\zeta - \eta_i|^\alpha$$

for $\zeta \in B_\delta(\eta_i) \cap \Omega$ and $\delta > 0$ small enough.

(ii) Consequently,

$$c|\zeta - \eta_i|^{\alpha-1} \leqslant |v'(\zeta)| \leqslant C|\zeta - \eta_i|^{\alpha-1}$$

for these ζ.

(iii) Obviously,

$$c(1 - |\zeta|)^{\alpha-1} \leqslant |v'(\zeta)| \leqslant C(1 - |\zeta|)^{\alpha-1} \text{ for } \zeta \in I_1 \cup I_2,$$
$$c|\zeta - \eta_i|^{\alpha-1} \leqslant |v'(\zeta)| \leqslant C|\zeta - \eta_i|^{\alpha-1} \text{ for } \zeta \in I.$$

Furthermore, we have

$$\int_{\partial\Omega} |\xi_n(\zeta) - 1|^2 |v'(\zeta)|\, |d\zeta| = \int_{\partial\Omega} (|\xi_n(\zeta)|^2 - 2\mathrm{Re}\, \xi_n(\zeta) + 1)|v'(\zeta)|\, |d\zeta|.$$

We start with the second term on the right-hand side

$$\int_{\partial\Omega} 2\mathrm{Re}\, \xi_n(\zeta)|v'(\zeta)|\, |d\zeta| = 2\mathrm{Re} \int_{\mathbb{T}} \xi_n(z)\, |dz|$$
$$= 4\pi\mathrm{Re}\, \xi_n(u(0)) = 4\pi\mathrm{Re}\, \xi_n(\zeta_0),$$

where $\xi_n(z) = \xi_n(u(z))$ and $|dz| = 2\pi dm(z) = d\theta,\ z = e^{i\theta}$. The last expression in the displayed formula tends to 4π by Theorem 0.3. Furthermore,

$$\int_{\partial\Omega} |v'|\, |d\zeta| = \int_{\mathbb{T}} |dz| = 2\pi$$

and it remains to show that

$$\lim_{n\to\infty} \int_{\partial\Omega} |\xi_n|^2 |v'|\, |d\zeta| \leqslant 2\pi. \tag{4.3}$$

We split the last integral into two integrals over I and $I_1 \cup I_2$, respectively. Then we obtain

$$\int_I |\xi_n|^2 |v'|\, |d\zeta| = \int_I |\varphi_n^*|^2 \sigma'_{\mathrm{ac}} |v'|\, |d\zeta| \leqslant 2\pi \int_I |\varphi_n^*|^2 |v'|\, d\sigma$$

and the last quantity tends to $\int_I |v'(\zeta)|\, |d\zeta|$ by (1.1).

We now turn to the integral over $I_1 \cup I_2$. Take any $\varepsilon > 0$ and freeze it. For any $\delta > 0$ (its choice will be made precise later)

$$\int_{I_1 \cup I_2} |\xi_n|^2 |v'|\, |d\zeta| = \int_{I_1 \cup I_2, |\zeta| \geqslant 1-\delta} \cdots + \int_{I_1 \cup I_2, |\zeta| < 1-\delta} \cdots$$

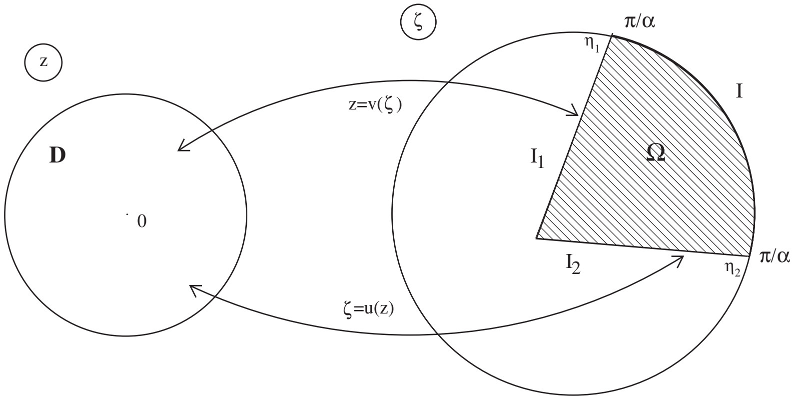

Fig. 1.

and we get for the first integral

$$\int_{I_1\cup I_2,|\zeta|\geqslant 1-\delta} |\xi_n(\zeta)|^2|v'(\zeta)|\,|d\zeta| \leqslant C\int_0^\delta \frac{1}{s}\,s^{\alpha-1}ds = C\int_0^\delta s^{\alpha-2}\,ds$$
$$= C\delta^{\alpha-1}.$$

Above, we used that $\alpha > 1$, a bound from (iii) and the inequality proved in Lemma 4.2. We pick δ small enough to satisfy $C\delta^{\alpha-1} < \varepsilon$.

Making $\delta > 0$ smaller, if necessary, we can guarantee that

$$\left|\int_{I_1\cup I_2,|\zeta|\geqslant 1-\delta} |v'|\,|d\zeta|\right| < \varepsilon.$$

Then, since ξ_n tends to 1 uniformly for $|\zeta| < 1-\delta$, we take n big enough to have

$$\left|\int_{I_1\cup I_2,|\zeta|<1-\delta} |\xi_n|^2|v'|\,|d\zeta| - \int_{I_1\cup I_2,|\zeta|<1-\delta} |v'|\,|d\zeta|\right| < \varepsilon.$$

Summing up the inequalities written above, we see that for a large n

$$\left|\int_{I_1\cup I_2} |\xi_n|^2|v'|\,|d\zeta| - \int_{I_1\cup I_2} |v'|\,|d\zeta|\right| < C\varepsilon$$

which shows

$$\lim_{n\to\infty}\int_{I_1\cup I_2} |\xi_n|^2|v'|\,|d\zeta| = \int_{I_1\cup I_2} |v'|\,|d\zeta|.$$

So, relation (4.3) is proved. Thus, we obtain

$$\lim_{n\to\infty}\int_I |\xi_n(\zeta)-1|^2|v'(\zeta)|\,|d\zeta| \leqslant \lim_{n\to\infty}\int_{\partial\Omega} |\xi_n(\zeta)-1|^2|v'(\zeta)|\,|d\zeta|$$
$$\leqslant 2\pi\lim_{n\to\infty} 2\mathrm{Re}(1-\xi_n(\zeta_0)) = 0$$

and the lemma is proved for any closed arc $I' \subset I$. $\square$

Remark 4.4. (i) The lemma also holds for a finite union $A = \cup I_k$, where I_k are closed arcs that do not contain points from $\{\zeta_k\}$.

(ii) For these arcs I, we also have

$$\lim_{n\to\infty} \int_I |\tilde{D}\tilde{\varphi}_n^*|^2 \, dm = m(I).$$

(iii) For $A \subset \mathbb{T}$ defined in (i),

$$\limsup_{n\to\infty} \int_{\mathbb{T}\setminus A} |\xi_n|^2 \, dm \leqslant m(\mathbb{T}\setminus A)$$

and here $\{\zeta_k\}$ necessarily lie in $\mathbb{T}\setminus A$.

To prove (ii) notice that $\| \|\xi_n\|_{L^2(I)} - \|1\|_{L^2(I)} | \leqslant \|\xi_n - 1\|_{L^2(I)}$, and the latter quantity tends to 0 as $n \to \infty$. As for (iii), we have

$$\lim_{n\to\infty} \int_A |\xi_n|^2 \, dm = \lim_{n\to\infty} \int_A |\varphi_n|^2 \sigma'_{\text{ac}} \, dm = m(A)$$

so

$$\begin{aligned}\limsup_n \int_{\mathbb{T}\setminus A} |\varphi_n|^2 \sigma'_{\text{ac}} \, dm &\leqslant 1 - \liminf_n \int_A |\varphi_n|^2 \sigma'_{\text{ac}} \, dm \\ &= 1 - \lim_{n\to\infty} \int_A |\varphi_n|^2 \sigma'_{\text{ac}} \, dm = 1 - m(A) = m(\mathbb{T}\setminus A).\end{aligned}$$

Proof of Theorem 0.4. The proof immediately follows from the Lemma 4.3 and the above remarks. Indeed, take an arbitrary $\varepsilon > 0$ and fix it. Then, choose $A = \cup I_k$ (see (iii), Remark 4.4) in a way that $m(\mathbb{T}\setminus A) < \varepsilon$. For n big enough

$$\int_{\mathbb{T}\setminus A} |\xi_n - 1|^2 \, dm < C\varepsilon.$$

On the other hand, by Lemma 4.3

$$\lim_{n\to\infty} \int_A |\xi_n - 1|^2 \, dm = 0$$

and the theorem follows. $\square$

Remark 4.5. We have

$$\lim_{n\to\infty} \int_{\mathbb{T}} |\varphi_n|^2 \, d\sigma_s = 0$$

for $\sigma \in$ (pS).

This is obvious, since $\lim_{n\to\infty} \int_{\mathbb{T}} |\varphi_n|^2 \sigma'_{\text{ac}} \, dm = 1$ and $\|\varphi_n\|_\sigma^2 = 1$.

5. Modified wave operators and a variational principle

Proof of Theorem 0.5. We mainly follow [16, Section 10.7]. Let us compute $\mathcal{F}^{-1}\tilde{\Omega}_+\mathcal{F}_0$ on the vectors of the form $\{z^l\}_{l\in\mathbb{Z}}$; the reasoning for $\mathcal{F}^{-1}\tilde{\Omega}_-\mathcal{F}_0$ is similar. Notice that $A_0^{(n)}, A_{2j,k}^{(n)} \in i\mathbb{R}$, $A_{2j+1,k}^{(n)} \in \mathbb{R}$ and so the operator $e^{W(\mathcal{C},2n)}$ is unitary. Let $J = \mathcal{F}^{-1}\mathcal{F}_0$. Recalling (3.1), we get

$$\begin{aligned}\mathcal{F}^{-1}\tilde{\Omega}_+\mathcal{F}_0 z^l &= \lim_{n\to+\infty} \mathcal{F}^{-1}e^{W(\mathcal{C},2n)}\mathcal{C}^n\mathcal{F}\,(\mathcal{F}^{-1}\mathcal{F}_0)\,\mathcal{F}_0^{-1}\mathcal{C}_0^{-n}\mathcal{F}_0 z^l\\ &= \lim_{n\to+\infty} e^{W(z,2n)}z^n J z^{-n}z^l = \lim_{n\to+\infty} e^{W(z,2(n+l))}z^{n+l}Jz^{-n}\\ &= z^l \lim_{n\to+\infty}\psi_{2(n+l)}(z)z^n J z^{-n}\end{aligned}$$

and, of course, all limits are to be understood in $L^2(\sigma)$-sense. We can assume $n \in \mathbb{Z}_+$ without loss of generality. Then, $\mathcal{F}_0 z^{-n} = \mathcal{F}_0\chi_{2n}^{(0)} = e_{2n}$ and $\mathcal{F}^{-1}e_{2n} = z^{-n}\varphi_{2n}^*(z)$. So

$$\begin{aligned}\lim_{n\to+\infty}\psi_{2(n+l)}(z)z^nJz^{-n} &= \lim_{n\to+\infty}\psi_{2(n+l)}(z)z^{n-n}\varphi_{2n}^*(z)\\ &= \lim_{n\to+\infty}\psi_{2n}\varphi_{2n}^* + \lim_{n\to+\infty}(\psi_{2(n+l)}-\psi_{2n})\varphi_{2n}^*. \qquad (5.1)\end{aligned}$$

We will prove a little later that

$$\lim_{n\to+\infty}(\psi_{2(n+l)}-\psi_{2n})\varphi_{2n}^* = 0 \qquad (5.2)$$

in $L^2(\sigma)$-sense. As for the first term in (5.1), we have

$$\lim_{n\to+\infty}\int_{\mathbb{T}}|\tilde{D}\psi_n\varphi_n^* - 1|^2\,dm = 0$$

by Theorem 0.4. This is the same as

$$\lim_{n\to+\infty}\int_{\mathbb{T}}\left|\psi_n\varphi_n^* - \frac{1}{\tilde{D}}\right|^2 d\sigma_{\rm ac} = 0$$

or, together with $\lim_{n\to+\infty}\int_{\mathbb{T}}|\varphi_n^*|^2\,d\sigma_s = 0$ (see Remark 4.5)

$$\lim_{n\to+\infty}\psi_n\varphi_n^* = \frac{1}{\tilde{D}}\chi_{E_{\rm ac}}$$

in $L^2(\sigma)$-sense, which is exactly the first relation in (0.10). Above, $E_{\rm ac} = \mathbb{T}\backslash\operatorname{supp}\sigma_s$. Let us prove relation (5.2). We have

$$\|(\psi_{2(n+l)}-\psi_{2n})\varphi_{2n}^*\|_\sigma^2 = \int_{\mathbb{T}}|\psi_{2n,2(n+l)}-1|^2|\varphi_n^*|^2\,d\sigma,$$

where

$$\psi_{2n,2(n+l)}(z) = \exp\left((A_0^{(2(n+l))} - A_0^{(2n)}) + \sum_k \sum_{j=1}^{2\kappa_k} (A_{j,k}^{(2(n+l))} - A_{j,k}^{(2n)}) \left\{ \frac{z+\zeta_k}{z-\zeta_k} \right\}^j \right).$$

Since, by (1.2), $\lim_{k\to\infty} \alpha_k = 0$ and the coefficients A_0^n, $A_{j,k}^{(n)}$ depend on a finite number of α_k only, we have that the expressions in the small round brackets above tend to zero as $n \to \infty$. Once again, take an arbitrary $\varepsilon > 0$ and fix it. Then, we choose arcs I'_k, $A' = \cup I'_k$, with the properties $m(A') < \varepsilon$ and $\{\zeta_k\} \subset A'$. By Remark 4.5 and (iii), Remark 4.4,

$$\int_{A'} |\psi_{2n,2(n+l)} - 1|^2 |\varphi_n^*|^2 \, d\sigma \leqslant 4 \int_{A'} |\varphi_n^*|^2 \, d\sigma < C\varepsilon$$

for n big enough. On the other hand, $\psi_{2n,2(n+l)}$ uniformly converges to 1 on $\mathbb{T}\backslash A'$. Hence,

$$\lim_{n\to+\infty} \int_{\mathbb{T}\backslash A'} |\psi_{2n,2(n+l)} - 1|^2 |\varphi_n^*|^2 \, d\sigma = 0$$

and (5.2) follows. $\square$

Below, we resort to the notation from the Introduction (see Theorem 0.6).

Proof of Theorem 0.6. We choose a constant C_0 from the condition $C_0 \int_{\mathbb{T}} p \, dm = 1$ and denote the polynomial $C_0 p$ by p_0. Similarly to Lemma 2.1, we have

$$p_0(t) = a_0 + 2\mathrm{Re} \sum_{j=1}^N a_j t^j$$

and $a_0 = (p_0, 1) = \int_{\mathbb{T}} p_0 \, dm = 1$. For any $g \in \mathcal{P}_0$ and $j \geqslant 1$, we have

$$\int_{\mathbb{T}} \log|g| \, dm = \log g(0), \qquad \int_{\mathbb{T}} \log|g| \, t^j \, dm = \frac{1}{2j!} \overline{(\log g)^{(j)}(0)}.$$

So

$$\log \lambda(g) = \log g(0) + \mathrm{Re} \sum_{j=1}^N \frac{\bar{a}_j}{j!} (\log g)^{(j)}(0)$$

or, what is the same

$$\lambda(g) = g(0) \, \exp\left(\mathrm{Re} \sum_{j=1}^N \frac{\bar{a}_j}{j!} (\log g)^{(j)}(0) \right).$$

Now, if $g \in \mathcal{P}'_1$, we have $1 = \lambda(g) = \|g\|_\sigma \, \lambda(f)$ with $f = g/\|g\|_\sigma \in \mathcal{P}_0$ and $\|f\|_\sigma = 1$. Consequently, $\|g\|_\sigma = \lambda(f)^{-1}$ for these g, f and the infimums in (0.11) are indeed equal.

For any $g \in \mathcal{P}_0$, $\|g\|_\sigma \leqslant 1$, the Jensen inequality implies

$$\exp\left(\int_{\mathbb{T}} p_0 \log \frac{|g|^2 \sigma'_{\mathrm{ac}}}{p_0}\, dm\right) \leqslant \int_{\mathbb{T}} |g|^2\, d\sigma \leqslant 1.$$

This means precisely that

$$\exp\left(\int_{\mathbb{T}} p_0 \log \frac{\sigma'_{\mathrm{ac}}}{p_0}\, dm\right) \leqslant \exp\left(-2\int_{\mathbb{T}} p_0 \log |g|\, dm\right) = \frac{1}{\lambda(g)^2}$$

and the first inequality in (0.11) is proved. To deal with the rest, recall that the measures $|\varphi_n^*|^{-2}\, dm$ converge weakly to $d\sigma$, and

$$\liminf_n \int_{\mathbb{T}} p_0 \log \frac{1}{|\varphi_n^*|^2}\, dm \leqslant \limsup_n \int_{\mathbb{T}} p_0 \log \frac{1}{|\varphi_n^*|^2}\, dm \leqslant \int_{\mathbb{T}} p_0 \log \sigma'_{\mathrm{ac}}\, dm$$

by the semicontinuity of entropy [16, Section 2.3]. The leftmost expression above is exactly $-2\log\lambda(\varphi_n^*)$, and we complete the proof of the theorem taking exponents in the last inequality. □

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